EXISTENCE AND CLASSIFICTION OF RADIAL SOLUTIONS OF A NONLINEAR NONAUTONOMOUS DIRICHLET PROBLEM

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Abstract. This paper generalizes a classification of solutions of a superlinear Dirichlet problem given in [13] to a nonautonomous case. In [12] the increasing of $f(t)$ was used to prove the classification and in [13] the unicity of the solution of the *Cauchy* problem was used. Here the classification appears as a consequence of the *a priori* estimates. It results that existence classificarion remain true for a class of nonautonomous problems.

1. Introduction

We are interested by radial solutions of the nonautonomous problem

$$-\Delta u = g(u) - \lambda - f(x), \text{ on } \Omega \text{ and } u = 0 \text{ on } \partial\Omega \tag{1}$$

where Ω denotes the unit ball in $\mathbf{R}^n$, $\lambda > 0$, f is a C^1 radial function on Ω. $g \in C^{0,\alpha}(\mathbf{R}, \mathbf{R})$ and there exists $A > 0$ such that $g_+ = g\big|_{[A,\infty[}$ is positive, increasing, differentiable and convex, $g_- = g\big|_{]-\infty,-A]}$ is positive, convex and decreasing. In addition

$$\lim \frac{g(x)}{x} = \pm\infty, \quad x \to \pm\infty \tag{2}$$

$$\lim \sqrt{\frac{R(x)}{x}}\, \frac{g_+^{-1}(x)}{g_-^{-1}(x)} = \pm\infty, \quad x \to \pm\infty \tag{3}$$

A classical problem of the existence of radial solutions still interesting in *superlinear* case see [6] and [10].

For the positone problem different methods have been used [10], and for the nonpositone problem, radial solutions have been considered using the shooting method [6][3]. Here we deal with the *nonpositone* problem using the homotopy of the topological degree [8].

P. L. Lions in [9] notes that many existence results of nodal solutions have been obtened but no classification of solutions have been given.

Remark that a classification of solutions set was introduced by P.H. Rabinowitz [11] based on the number of zeros of the solution $u(t)$ to prove existence results for a semilinear Sturm-liouville problem.

In this paper we use the homotopy of the topological degree and a classification of solutions based on the number of zeros of the second hand side of Eq.(1) $g(u(t)) - \lambda - f(t) = 0$, $t \in \mathbf{R}$. This approch represents an alternative for the shooting method and have been used in [12][13].

This paper generalize the existence result given in [13] for a nonautonomous case. The main result is the Proposition(3) in which the classification of solutions set appears as a consequence of the *a priori* estimates. Indeed in [12] the classification was given by the increasing property of $f(t)$, see proof of Proposition(3) Eq.(2.19), and in [13] the unicity of the solution of the *Cauchy* problem was used, see Proposition(4) [13].

Remark that the topological method is not limited by *critic Pohozaev-Sobolev exponent* but only by *a priory* estimates. Hence, the existence result given in Theorem (1) [13] depends only on conditions (2) and (3) and stills valid for $\mathbf{R^n}$, $n \geq 1$. To our knowledge the most general existence results known at this time for nodal solutions of *nonpositone* Elliptic problems are subject to the limite of *critic Pohozaev-Sobolev exponent*.

A remarkable *a priori* estimates for positive solutions of elliptic problems was given in [7] and used to get existence result with the topological degree.

Here, properties of the *nonpositone* problem and nodal solutions have been exploted to get an *a priori* estimates which is not limited by the *critic Pohozaev-Sobolev exponent*.

The plane of the proof is similar to [13] and most arguments of proofs remain true for (1). So we will give details just for the proof of Proposition(3) which generalizes Proposition(4) in [13].

2. Existence and classification of solutions

We consider the problem

$$-u''(t) - \frac{n-1}{t}u'(t) = g(u(t)) - \lambda - f(t) \tag{4}$$

$$u'(0) = 0, u(1) = 0$$

u having a local minimum in *zero*. This is a non autonomous problem related to (5) in [13]. In addition suppose that $f \in C^1([0,1], \mathbf{R})$.

Recall that $\lambda > 0, g \in C(\mathbf{R}, \mathbf{R})$ and there exists $A > 0$ such that $g_+ = g\big|_{[A,\infty[}$ is positive, increasing, differentiable and convex, $g_- = g\big|_{]-\infty,-A]}$ is positive, convex and decreasing. In addition

$$\lim \frac{g(x)}{x} = \pm\infty, \quad x \to \pm\infty \tag{5}$$

Let $k \in N$, $\lambda > A$, $E = \{u \in C^1([0,1], \mathbf{R}) : u'(0) \leq 0, u(1) = 0\}$ and $Z_k(\lambda)$ a subset of E defined by

$$Z_k(\lambda) = \left\{u \in E : u(t) - g_+^{-1}(\lambda + f(t)) \text{ has } k \text{ simple zeros in } [0,1]\right\}$$

We denote $M = \|f\|_{C^1}$.

The following proposition recalls the *a priori* estimate given in proposition (2) in [13].

Proposition. There exist $C > 0$ and $K(\lambda)$ a continuous function defined on $[C, \infty[$ such that, for each solution (u, λ) of (1) satisfying $\lambda > C$ and $u'(0) \leq 0$, we have $\|u\| < K(\lambda)$.

For a local maximum β

$$u(\beta) < 2R(4(\lambda + M)), \quad R(x) = \max\{|g_-^{-1}(x)|, |g_+^{-1}(x)|\}$$

and for a local minimum α

$$|u(\alpha)| \leq R(\lambda + M)$$

The proof of the propostion is the same as proof of Proposition(2) in [13].

Some general formulas.

— The mean theorem gives

$$\left|\int_{g_+^{-1}(\lambda+m_1)}^{g_+^{-1}(\lambda+m_2)} (g(u) - \lambda)du\right| \leq m_2 \left|g_+^{-1}(\lambda + m_2) - g_+^{-1}(\lambda + m_1)\right|$$

and gives $\mu \in]m_1, m_2[$

$$g_+^{-1}(\lambda+m_2) - g_+^{-1}(\lambda+m_1) = \frac{m_2 - m_1}{g'(g_+^{-1}(\lambda+\mu))}$$

$$\left|\int_{g_+^{-1}(\lambda+m_1)}^{g_+^{-1}(\lambda+m_2)} (g(u)-\lambda)du\right| \leq m_2 \left|\frac{m_2 - m_1}{g'(g_+^{-1}(\lambda+\mu))}\right| \tag{6}$$

— for $x > a$ large enough g_+ is convex then

$$g'(x) > \frac{g(x)-g(a)}{x-a}$$

for x large enough there exists $\gamma > 0$ such that

$$g'(x) > \gamma g(x)/x$$

set $x = g^{-1}(\lambda+\mu)$ to get

$$\frac{1}{g'(g_+^{-1}(\lambda+\mu))} \to 0, \lambda \to +\infty \tag{7}$$

— Let $a, b \in [0,1]$

$$\begin{aligned} \int_a^b fu'dt &= (f(b)u(b) - f(a)u(a)) + \int_a^b f'udt \\ \left|\int_a^b fu'dt\right| &\leq 3M \max|u(t)| \quad \leq 6M\, R(4(\lambda+M)) \end{aligned} \tag{8}$$

— The concavity of g_+^{-1} implies that for $x > \alpha$ large enough

$$\frac{g_+^{-1}(x) - g_+^{-1}(\alpha)}{x-\alpha}$$

is decreasing, then for $b > a > 0$ and λ large enough

$$\frac{g_+^{-1}(b\lambda) - g_+^{-1}(\alpha)}{b\lambda - \alpha} < \frac{g_+^{-1}(a\lambda) - g_+^{-1}(\alpha)}{a\lambda - \alpha}$$

we deduce that there exists $\gamma > 0$ such that

$$g_+^{-1}(b\lambda) < \gamma g_+^{-1}(a\lambda) \tag{9}$$

Remark 1. *Increasing of g gives, for λ large enough, $u > g_+^{-1}(\lambda+f)$ implies $g(u) - (\lambda+f) > 0$, and $u = g_+^{-1}(\lambda+f)$ implies $g(u) - (\lambda+f) = 0$, hence $0 \leq u < g_+^{-1}(\lambda+f)$ implies $g(u) - (\lambda+f) < 0$.*

For β a local maximum $g(u(\beta)) - (\lambda + f(\beta)) \geq 0$, from which $g_+^{-1}(\lambda + f(\beta)) \leq u(\beta)$. (contrapositive of the last implication)

For α a positive local minimum $g_+^{-1}(\lambda + f(\alpha)) \geq u(\alpha)$.

The following lemma genaralizes Lemma(5) in [13] which is used in the following to prove Proposition(3).

Estimation of the derivative at zeros of $u(t) - g_+^{-1}(\lambda + f(t))$.

Lemma 2. *There exists a sequence (A_k) $(k \geq 1)$ of positive numbers such that, for each solution (u, λ) of (1) satisfying $u'(0) \leq 0$, $\lambda > A_k$ and $u - g_+^{-1}(\lambda+f)$ having at least k zeros, there exist $B > 0$ satisfying for the k largest zeros*

$$|u'(\tau)| > B\sqrt{\lambda g_+^{-1}(\lambda/2)}$$

Proof. *The k largest zeros of $u - g_+^{-1}(\lambda + f)$ are denoted by $\tau_1 < \tau_2 < .. < \tau_k < 1$. τ_k represents the largest zero.*

Estimation of $u'(\tau_k)$.

Let $\eta \in]\tau_k, 1[$ be the smallest zero of $u(t)$. Since $g_+^{-1}(\lambda + f) > u$, from remark(1) u has no local maximum in $]\tau_k, \eta[$, then it is decreasing on $[\tau_k, \eta]$ and from (1) it is convex.

Let a be the unique element of $]\tau_k, \eta[$ such that $u(a) = g_+^{-1}(\lambda/2)$. Denoting by $h(t)$ the segment joining $u(a)$ and $u(\eta) = 0$, and setting $v(t) = h(t) - u(t)$ on $]a, \eta[$, then $-v'' = \lambda + f - g(u) - pu'$. Since $u < g_+^{-1}(\lambda/2)$ and is decreasing $-v'' > \lambda/2$, since $v < g_+^{-1}(\lambda/2)$

$$-v'' > \frac{\lambda}{2g_+^{-1}(\lambda/2)} v, \text{ on }]a, \eta[$$

$$v(\eta) = v(a) = 0$$

setting $t = s\,(\eta - a) + a, s \in [0, 1]$ and $w(s) = v(s\,(\eta - a) + a)$

$$-w'' > (\eta - a)^2 \frac{\lambda}{2g_+^{-1}(\lambda/2)} w, \text{ on }]0, 1[$$

$$w(0) = w(1) = 0$$

the comparison theorem of Sturm gives $(\eta - a) < \sqrt{2}\pi\sqrt{g_+^{-1}(\lambda/2)/\lambda}$.

Since u is convex on $]\eta, \tau_k[$, $|u'(\tau_k)| > |u'(a)| > u(a)/(\eta - a)$, hence $|u'(\tau_k)| > \frac{1}{\sqrt{2}\pi}\sqrt{\lambda g_+^{-1}(\lambda/2)}$.

We shall use the recurrence argument.

Let $B > 0$, $\delta > 0$, τ_i and τ_{i+1} two consecutive zeros such that $|u'(\tau_{i+1})| > B\sqrt{\lambda g_+^{-1}(\lambda/2)}$, then for λ large enough we have $|u'(\tau_i)| > (B - \delta)\sqrt{\lambda g_+^{-1}(\lambda/2)}$.

Indeed, multiplying (1) by u' and integrating to get

$$\frac{u'^2(\tau_i)}{2} \geq \frac{u'^2(\tau_{i+1})}{2} + \int_{u(\tau_i)}^{u(\tau_{i+1})} (g(u) - \lambda) du - \int_{\tau_i}^{\tau_{i+1}} f u' dt$$

then (3,5) give

$$\frac{u'^2(\tau_i)}{2} \geq \frac{B^2}{2} \lambda g_+^{-1}(\lambda/2) - M \left| \frac{f(\tau_{i+1}) - f(\tau_i)}{g'(g_+^{-1}(\lambda))} \right| - 6M\, R(4(\lambda + M))$$

(2,6) give $\frac{R(4(\lambda+M))}{\lambda g_+^{-1}(\lambda/2)} \to 0$ and (4) gives $\frac{f(\tau_{i+1})-f(\tau_i)}{g'(g_+^{-1}(\lambda))} \to 0$. □

The following proposition genaralizes the Proposition(4) in [13].

Proposition 3. *There exists a sequence $(B_k)(k \geq 0)$ of positive numbers such that, for each $\lambda > B_k$, (1) has no solution $u \in \partial Z_{2k}(\lambda)$ satisfying $u'(0) \leq 0$.*

Proof. By contradiction, let $u \in \partial Z_{2k}(\lambda)$ be a solution of (1).

Case $k = 0$: Let (v_n) a sequence of solutions in $Z_0(\lambda)$ such that $v_n \to u$. From remark(1) $v_n < 0$ on $]0, 1[$ then $u \leq 0$, hence $u - g_+^{-1}(\lambda + f)$ has no zero from which $u \in Z_0(\lambda)$ thus $u \notin \partial Z_0(\lambda)$, contradiction.

Case $k \geq 1$: *First, we shall prove that $u - g_+^{-1}(\lambda + f)$ has at most $2k$ simple zeros.*

Indeed, let τ be a simple zero, then there exist $\epsilon_0 > 0$, $\epsilon_1 > 0$ and $\delta > 0$ such that τ is the unique zero on $]\tau - \epsilon_0, \tau + \epsilon_0[$, (one assume that $u - g_+^{-1}(\lambda + f)$ is

increasing. If it is decreasing the inequalities are inverse and the proof is similar)

$$\left| \begin{array}{l} u(\tau-\epsilon_0) - g_+^{-1}(\lambda + f(\tau-\epsilon_0)) < -\epsilon_1 \\ u(\tau+\epsilon_0) - g_+^{-1}(\lambda + f(\tau+\epsilon_0)) > \epsilon_1 \\ u' - \left[g_+^{-1}(\lambda+f)\right]' > \delta \end{array} \right.$$

Let (v_n) be a sequence of $Z_{2k}(\lambda)$ such that $v_n \to u$ in E, there exists $n(\epsilon_0, \epsilon_1, \delta) \in N$ such that for $n > n(\epsilon_0, \epsilon_1, \delta)$

$$\left| \begin{array}{l} |u(\tau-\epsilon_0) - v_n(\tau-\epsilon_0)| < \epsilon_1/2 \\ |u(\tau+\epsilon_0) - v_n(\tau+\epsilon_0)| < \epsilon_1/2 \\ \|u' - v_n'\|_\infty < \delta/2 \end{array} \right.$$

from which

$$\left| \begin{array}{l} v_n(\tau-\epsilon_0) - g_+^{-1}(\lambda + f(\tau-\epsilon_0)) < -\epsilon_1/2 \\ v_n(\tau+\epsilon_0) - g_+^{-1}(\lambda + f(\tau+\epsilon_0)) > \epsilon_1/2 \\ v_n' - \left[g_+^{-1}(\lambda+f)\right]' > \delta/2 \end{array} \right.$$

which implies that $v_n - g_+^{-1}(\lambda+f)$ has a unique simple zero on $]\tau-\epsilon_0, \tau+\epsilon_0[$. Since $v_n \in Z_{2k}(\lambda)$, $v_n - g_+^{-1}(\lambda+f)$ has exactly $2k$ simple zeros, then $u - g_+^{-1}(\lambda+f)$ has at most $2k$ simple zeros.

There are not exactly m simple zeros with $m < 2k$.

Indeed, by contradiction assume that $u \in Z_m(\lambda)$. Since $Z_{2k}(\lambda)$ and $Z_m(\lambda)$ are open sets of E and $Z_m(\lambda) \cap Z_{2k}(\lambda) \neq \emptyset$, then $Z_m(\lambda) \cap \partial Z_{2k}(\lambda) = \emptyset$, contradiction.

Last, since there exist at most $2k$ simple zeros of $u - g_+^{-1}(\lambda+f)$ there exists τ_j a zero which is not simple $j \leq 2k+1$. From the lemma (2), there exists $A_{2k+1} > 0$ such that for $\lambda > A_{2k+1}$ $\quad |u'(\tau_j)| > B\sqrt{\lambda g_+^{-1}(\lambda/2)}$

On the other hand $[g_+^{-1}(\lambda+f)]' = \frac{f'}{g_+'(g_+^{-1}(\lambda+f))}$, (4) implies that $|u'(\tau_j)| > \left|\left(g_+^{-1}(\lambda+f(\tau_i))\right)'\right|$ for λ large enough, then τ_j is a simple zero of $u - g_+^{-1}(\lambda+f)$, contradiction. $\square$

References

[1] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983) 437–477.

[2] A. Capozzi, D. Fortunato, G. Palmieri, An existence result for nonlinear elliptic problems involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire 2 (1985) 463–470.

[3] A.Castro and R. Shivaji, *Multiple Solutions for a Dirichlet Problem with Jumping Nonlinearities.* II J. M. A. and Appl 133, 505-528 (1988).

[4] G. Cerami, D. Fortunato, M. Struwe, Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 341–350.

[5] G. Cerami, S. Solinini, M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents, J. Funct. Anal. 69 (1986) 289–306.

[6] W. Dambrosio: *On the multiplicity of radial solutions to superlinear Dirichlet problems in bounded domains,* J. Diff. Equat. 196 (2004) 91-118.

[7] D.G.de Figueiredo, P.L. Lions, R.D. Nussbaum : A priori estimats and existence results of positive solutions of semilinear elliptic equations, J. Math. Pures. Appl,

[8] J.Leray et J.Schauder: *Topologie et équations fonctionnelles.* Ann Sci. Ecole Norm Sup Vol. 3, 51, 1934, p 45-414.

[9] P. L. Lions: *On the existence of positive solutions of semilinear elliptic equations*, SIAM review, Vol. 24, (1982)

[10] Massimo Grossi : Nodal solutions for an elliptic problem involving large nonlinearities. J. Differential Equations 245 (2008) 2917–2938.

[11] P. H. Rabinowitz: *Nonlinear Sturm-Liouville Problems for Second order ordinary differential equations,* Communications on pure and applied mathematics, vol XXIII (1970), 939-961.

[12] M. Rouaki: *Topological Degree and a nonlineaar Dirichlet problem.* Nonlinear Analysis MTA 54 (2003) 801-817.

[13] M. Rouaki: *Nodal radial solutions for a superlinear problem.* Nonlinear Analysis RWA 8 (2007) 563-571.

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